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A Note on the Quadratic Divergence in Hybrid Regularization

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Abstract

We consider the quadratic divergence of the Yang-Mills theory when we use the hybrid regularization method consisting of the higher covariant derivative terms and the Pauli-Villars fields. By the explicit calculation of the diagrams, we show that the higher derivative terms for the ghost fields are necessary for the complete cancellation of the quadratic divergence.

1 Introduction

An invariant regularization scheme is necessary for the treatment of the ultra-violet divergence in quantum gauge theories. The dimensional regularization is known as the most powerful and popular method, but it is not available for the theory preserving the symmetry which depends on the space-time dimension, like chiral gauge theory or topological field theory. In such a case, the hybrid regularization based on the higher covariant derivative (HCD) method [1, 2, 3] is expected to be useful.

The HCD method is a partial regularization itself because the higher derivative terms are introduced in a covariant way; HCD terms render the propagators less divergent but the vertices more divergent. It is easily shown by the calculation of the superficial degree of divergence that some diagrams at one-loop level are left unregularized. We have to introduce an additional regularization scheme to regularize the divergence from these diagrams.

The Pauli-Villars (PV) regularization is suitable for the additional regularization from the standpoint of the invariant method, because the PV regulators are constructed in a chiral invariantly [4, 5] or parity invariantly [6, 7]

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by an infinite number of them. Applying such PV fields to the hybrid regularization, the method is expected to be an invariant regularization scheme that is available for the theory containing higher symmetries like the supersymmetric gauge theories. It has not been verified, however, whether such special PV fields actually regularize the intrinsically divergent theory in the framework of the hybrid regularization: at least the Yang-Mills (YM) theory must be regularized properly.

In recent years, on the other hand, it was pointed out that the original Slavnov's hybrid regularization scheme does not give the correct value of the coefficient of the renormalization group (RG) β -function when the YM theory is regularized by this scheme [8]. Nevertheless, this problem was overcome by the minor modification of the scheme [9, 10], and it was confirmed that the modification is proper at the one-loop level by an explicit calculation [11].

Since only the logarithmic divergence plays an important role in the calculation of the RG functions, the other divergence, the quadratic divergence in the case of four dimensions, has not been seriously considered. If the regularization works properly, the quadratic divergence ought to be canceled out. But in this scheme, the cancellation is not trivial because the HCD term contributes to the quadratic divergence and then increases the complexity of the divergence [11]. So it is worthy to show the cancellation of the quadratic divergence for the consistency of the hybrid regularization scheme.

In this note, we confirm that the quadratic divergence is actually canceled out in the YM theory with the hybrid regularization of the HCD and the PV method. By an explicit calculation of the vacuum polarization tensor, it is shown that the higher derivative terms for the ghost fields are necessary for the complete regularization of this method. Using an infinitely many PV fields as the additional regulator, we also check the consistency of such PV fields in the quadratic divergence when they are used in the hybrid regularization.

2 The regularization method

We consider the $SU(N)$ Yang-Mills theory in four dimensional Euclidean space-time. The action is given by

$$S = S_{\text{YM}} + S_{\text{GF}}, \quad (1)$$

where

$$S_{\text{YM}} = \frac{1}{4} \int d^4x F_{\mu\nu}^a F^{\mu\nu a}, \quad (2)$$

$$S_{\text{GF}} = \int d^4x \left[\frac{\xi_0}{2} b^a b^a - b^a (\partial^\mu A_\mu^a) + \bar{c}^a (\partial_\mu D^\mu c)^a \right], \quad (3)$$

with the field strength $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ and the covariant derivative $D_\mu^{ac} = \delta^{ac} \partial_\mu + g f^{abc} A_\mu^b$. Here A_μ^a , c^a , \bar{c}^a and b^a denote the gauge field, ghost, anti-ghost and auxiliary field respectively, ξ_0 is the gauge-fixing parameter and f^{abc} is the structure constant of the gauge group.

The hybrid regularization consists of the following two steps: first we introduce HCD terms and next PV fields. The HCD terms improve the behavior of propagators at large momentum, rendering the theory less divergent at the cost of the emergence of new vertices, and the theory is reduced to superrenormalizable, i.e. there are just a finite number of divergent loops. As see later, all the diagrams except one-, two-, three- and four-point functions at one-loop level are convergent with a suitable choice of the HCD action. We deal with the remaining divergence by a PV type of regularization.

2.1 Introduction of HCD terms

We first regularize the action by an addition of the HCD term;

$$S_\Lambda = S_{\text{YM}} + S_{\text{HCD}} + S_{\text{GF}}^H. \quad (4)$$

S_{HCD} is the HCD action and S_{GF}^H is the modified gauge-fixing action when we use the HCD method. The explicit forms are

$$S_{\text{HCD}} = \frac{1}{4\Lambda^4} \int d^4x (D^2 F_{\mu\nu})^a (D^2 F^{\mu\nu})^a, \quad (5)$$

$$S_{\text{GF}}^H = \int d^4x \left[\frac{\xi_0}{2} b^a b^a - b^a H(\partial^2/\Lambda^2) (\partial^\mu A_\mu^a) + \bar{c}^a H(\partial^2/\Lambda^2) (\partial_\mu D^\mu c)^a \right], \quad (6)$$

where Λ is a cutoff parameter which has mass dimension of one, and $H(\partial^2/\Lambda^2)$ is a dimensionless function and must be a polynomial of ∂^2/Λ^2 to ensure the locality of the gauge field. Its explicit form is determined by the behavior of the gauge propagator which is obtained from (4) as follows:

$$\frac{\Lambda^4}{p^4(p^4 + \Lambda^4)} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) + \frac{\xi_0}{p^4 H^2(p^2/\Lambda^2)} p_\mu p_\nu. \quad (7)$$

The first term has the momentum degree of -6 , so the second term must be the same degree or less to ensure the convergence of the diagrams; H^2

behaves $\sim p^4$ or higher at large p . While the familiar form of the propagator must be recovered in the limit of $\Lambda \rightarrow \infty$; H^2 converges to unity at large Λ . On these conditions, the simplest choice of the H^2 in momentum space is

$$H^2 \left(\frac{p^2}{\Lambda^2} \right) = 1 + \frac{p^4}{\Lambda^4}. \quad (8)$$

Since the first and the second term of (7) has the same denominator in this choice of H^2 , when we work in the Feynman gauge ($\xi_0 = 1$), the gauge propagator is reduced to

$$\frac{\Lambda^4}{p^2(p^4 + \Lambda^4)} \delta_{\mu\nu}. \quad (9)$$

We take the Feynman gauge in the explicit calculation of diagrams in the following section.

The HCD action is generally introduced in the form of $\frac{1}{4\Lambda^{2n}} \int d^4x (D^n F_{\mu\nu})^2$. In such a case, the superficial degree of divergence is written

$$\omega = 4 - 2n(L - 1) - E_A \quad (10)$$

where L and E_A are the number of loops and the number of the external lines of the gauge field. For all the diagrams higher than two-loops ($L \geq 2$), $n \geq 2$ always gives negative ω . This means that we may remove the higher loops by a suitable choice of n . For one-loop ($L = 1$), ω is not always negative for any n . As we take the higher n , though the propagator becomes more convergent at large momentum, the vertices are more divergent and increase their complexities. Then the calculation of the diagram is more complex even though at one-loop level. The most economical choice is $n = 2$ which leads the HCD action (5).

2.2 Introduction of Pauli-Villars fields

So far, all the diagrams except one-, two-, three- and four-point functions at one-loop level are convergent by the HCD action. We deal with the remaining divergence by a PV type of regularization.

Consider the following generating functional:

$$Z[J, \chi, \eta, \bar{\eta}] = \int \mathcal{D}A_\mu \mathcal{D}b \mathcal{D}\bar{c} \mathcal{D}c \exp[-S_\Lambda - S_J] \\ \prod_{j=1}^{\infty} \left[\det^{-\frac{\alpha_j}{2}} \mathbf{A}_j \right] \left[\det^{-\frac{\alpha_{-j}}{2}} \mathbf{A}_{-j} \right] \prod_{i=1}^{\infty} \left[\det^{\gamma_i} \mathbf{C}_i \right] \left[\det^{\gamma_{-i}} \mathbf{C}_{-i} \right], \quad (11)$$

where S_J is a source term consisting from J , χ , η and $\bar{\eta}$ which is the source of A_μ , b , \bar{c} and c respectively. $\left[\det^{-\frac{\alpha_j}{2}} \mathbf{A}_j\right]$ and $\left[\det^{\gamma_i} \mathbf{C}_i\right]$ are PV determinants for the gauge and ghost field respectively, defined by

$$\left[\det^{-\frac{\alpha_j}{2}} \mathbf{A}_j\right] = \int \mathcal{D}A_{j\mu} \mathcal{D}b_j \exp[-S_{M_j} - S_{b_j}], \quad (12)$$

$$\left[\det^{\gamma_i} \mathbf{C}_i\right] = \int \mathcal{D}\bar{c}_i \mathcal{D}c_i \exp[-S_{m_i}], \quad (13)$$

where $\{\alpha_j\}$ and $\{\gamma_i\}$ are real parameters to be fixed for each indices and the explicit forms of the functions in the exponents are

$$S_{M_j} = \frac{1}{2} \int d^4x d^4y \quad A_{j\mu}^a(x) \left[\frac{\delta^2 S_\Lambda}{\delta A_\mu^a(x) \delta A_\nu^b(y)} - M_j^2 \delta^{ab} g^{\mu\nu} \delta(x-y) \right] A_{j\nu}^b(y), \quad (14)$$

$$S_{b_j} = \int d^4x \left[\frac{\xi_j}{2} b_j^a b_j^a - b_j^a \tilde{H}(D^\mu A_{j\mu}^a) \right], \quad (15)$$

$$S_{m_i} = \int d^4x \left[\bar{c}_i^a \tilde{H}(D_\mu D^\mu c_i)^a - m_i^2 \bar{c}_i^a c_i^a \right]. \quad (16)$$

The field $A_{j\mu}^a$ is a PV field for the gauge of mass M_j , b_j^a an auxiliary field for $A_{j\mu}^a$, \bar{c}_i^a and c_i^a PV fields for the ghost and anti-ghost of mass m_i . We introduce a gauge-fixing parameter ξ_j for the correct regularization of the theory [9]. \tilde{H} is the HCD term for the ‘gauge-fixing function’ for the PV fields, which has the form

$$\tilde{H} = \left(1 + \frac{D^4}{\Lambda^4} \right)^{\frac{1}{2}}. \quad (17)$$

These PV fields have the same form as the ones that we used in the Chern-Simons gauge theory [7, 12]. The idea is to regularize the theory with the pairs of the two types of the PV fields A_j and A_{-j} . In the Chern-Simons gauge theory to construct the parity invariant regulator, the two fields are related by the parity transformation and represented by the slightly different actions, but in this case they have the same action except the sign of the index.

The reason why we have to introduce an infinite number of the PV fields comes from the idea ‘to regularize with the pair’. Imagine when the gauge field is regularized by the PV pairs which we prepared above. Since the number of the gauge field is one, introducing one pair corresponds to subtracting double the divergence. Then to remedy the over subtraction we

introduce another pair of opposite statistics which means adding double the divergence. To remedy the over addition we have to introduce the third pair. Such steps correspond to introducing fermionic PV fields ($\alpha_j = -1$) and bosonic PV fields ($\alpha_j = +1$) alternately. We repeat such steps alternately until the divergence is removed. Namely, we cannot regulate the theory by a finite number of PV pairs, but we need an infinite number. Then we take the following as PV conditions:

$$M_j = M|j|, \quad \alpha_j = (-1)^j. \quad (18)$$

In the same way, we take the PV conditions for ghost and anti-ghost such as $m_i = m|i|$ and $\gamma_i = (-1)^i$.

The generating functional (11) is invariant under the BRST transformations in the reference [7], and this regularization manifestly preserves BRST invariance.

2.3 Feynman rules

The regularized action S_Λ is decomposed into the kinetic part K and the vertex part V as follows [11, 13]:

$$\int d^4x \Psi(x)(K + V + M^2)\Phi(x), \quad (19)$$

where $\Psi(x)$ and $\Phi(x)$ denotes an arbitrary field and M its mass term. Since K and V consist of the Λ -free part from the YM term (we denote with suffix ‘0’) and the Λ -dependent part from HCD term (with suffix ‘ Λ ’) we formally decompose

$$K = K_0 + \frac{1}{\Lambda^4} K_\Lambda, \quad V = V_0 + \frac{1}{\Lambda^4} V_\Lambda. \quad (20)$$

Then the propagators are written in the form

$$\frac{1}{K + M^2} = \frac{1}{K_0 + M^2} \left(1 - \frac{K_\Lambda}{K_0 + M^2} \Lambda^{-4} + O(\Lambda^{-8}) \right). \quad (21)$$

Using this decomposition, the Feynman rules are written by the order of Λ and we calculate the quantum corrections order by order of Λ .

3 One-Loop Contributions

Now we calculate the one-loop vacuum polarization tensor order by order in Λ up to Λ^{-4} . There are eleven diagrams in Λ^0 order and twelve diagrams

in Λ^{-4} order. Each diagram has the quadratic divergence which must cancel in totally. We consider this divergence calculating each diagram under the Feynman gauge $\xi_j = 1$. The calculation is carried out under the same rules that we take in references [7, 12, 14] whose summaries are the following:

1. Take the same assignment for the internal momentum among the graphically same diagrams.
2. Take the infinite sum of the graphically same diagrams under the PV condition (18).
3. If there is no massless term (whose index j or i is zero) for the infinite sum, add the lacking terms and subtract the same ones to balance.

All the diagrams are classified into three groups by the kind of the internal line in the diagram and calculation is carried out in each group.

First we consider the diagrams that contains only $A_{j\mu}$ fields in the internal lines. The index j runs from $-\infty$ to ∞ except zero for these diagrams, if we take the infinite sum from $-\infty$ to ∞ we have to add the diagrams which contains ' $A_{0\mu}$ ' field in the internal lines. Fortunately, the diagrams in which the gauge field runs have the same structure of them and we take the infinite sum from $-\infty$ to ∞ without any extra terms. Then the total of the quadratic divergence from these diagrams is calculated

$$\frac{g^2 c_v \delta^{ab}}{8\pi^2} \left(\frac{M^2}{40} C_2 - \frac{9M^6}{154\Lambda^4} C_4 \right) \delta_{\mu\nu}. \quad (22)$$

Where C_2 and C_4 are the constants arising from the infinite sum of the index j [12, 14].

For the diagrams that contains the b_j fields, there is no diagrams from the field b to compensate the diagrams consisting of b_0 because any vertex of the form $\langle A_\mu A_\nu b \rangle$ does not exist in the theory. So we have to add the b_0 diagrams to take the infinite sum and subtract the same after the summations to balance out. The total contributions are calculated

$$\begin{aligned} & - \frac{g^2 c_v \delta^{ab}}{8\pi^2} \left(\frac{M^2}{40} C_2 - \frac{9M^6}{154\Lambda^4} C_4 \right) \delta_{\mu\nu} \\ & + g^2 c_v \delta^{ab} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 (k-p)^2} \left[2k^2 \delta_{\mu\nu} - 3k_\mu k_\nu + \frac{1}{\Lambda^4} (2k^6 \delta_{\mu\nu} - 4k^4 k_\mu k_\nu) \right]. \end{aligned} \quad (23)$$

The first line comes from the infinite sum with index j and the second line is the counter terms which we introduced to take the sum.

The situation does not alter in the diagrams containing \bar{c}_i and c_i in the internal lines. Since there are some differences in the vertices between ghost fields and PV fields for ghosts, \bar{c} and c do not play the role of \bar{c}_0 and c_0 . Then we have to add some massless terms for the infinite sums and subtract the same ones in the same way as the diagrams with b_j . We calculate the contributions from these diagrams as follows:

$$-g^2 c_v \delta^{ab} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(k-p)^2} \left[2k^2 \delta_{\mu\nu} - 3k_\mu k_\nu + \frac{1}{\Lambda^4} (2k^6 \delta_{\mu\nu} - 4k^4 k_\mu k_\nu) \right]. \quad (24)$$

The each contribution has the quadratic divergence proportional to the constant C_2 and C_4 after the infinite sum with indices, but these divergence cancel out in total within this group. Then only the quadratic divergence from the counter terms remain as in (24).

It is easy to see from (22), (23) and (24) that the quadratic divergence of the vacuum polarization tensor disappears from the theory.

Here we notice that the function $f(\partial^2/\Lambda^2)$ in the reference [8] does not cancels the quadratic divergence completely. In that reference, since the modified gauge-fixing action is inserted through the form $\frac{\xi_0}{2f} b^2 - b \partial^\mu A_\mu$ instead of (6), so the naively extended gauge-fixing action for the PV field is written by $\frac{\xi_j}{2f} b_j^2 - b_j \partial^\mu A_{j\mu}$. This action, however, breaks the BRST invariance because f contains usual derivative ∂_μ . This symmetry breaking effects to the cancellation of the quadratic divergence. In such a case, all the Feynman rules related to the auxiliary, ghost and their PV fields are modified in Λ^{-4} order, and then the quadratic divergence corresponds to (23) and (24) are calculated as follows:

$$-\frac{g^2 c_v \delta^{ab}}{8\pi^2} \left(\frac{M^2}{40} C_2 - \frac{6M^6}{154\Lambda^4} C_4 \right) \delta_{\mu\nu} + g^2 c_v \delta^{ab} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(k-p)^2} [2k^2 \delta_{\mu\nu} - 3k_\mu k_\nu], \quad (23')$$

$$-g^2 c_v \delta^{ab} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(k-p)^2} [2k^2 \delta_{\mu\nu} - 3k_\mu k_\nu]. \quad (24')$$

This result shows that the complete cancellation at Λ^{-4} does not occur though the contribution at Λ^0 cancels. We may choose the higher *covariant* derivative function \tilde{f} instead of f to avoid this difficulty as we extended H to \tilde{H} . If we substitute $\tilde{f} = 1 + \frac{D^4}{\Lambda^4}$ in place of $f = 1 + \frac{\partial^4}{\Lambda^4}$, some new diagrams arise from the new vertices among b_j under the naively treatment

of \tilde{f} . These diagrams, however, give no quadratic divergence to (24') in total, and then the divergence is not removed. This failure may be caused by the treatment of the non-local contribution in the function \tilde{f} . We may have to consider the exact treatment of such terms when we start from the action described by f .

4 Conclusion

In this note we consider the cancellation of the quadratic divergence of the YM theory regularized by the hybrid regularization consisting of the HCD method and the infinitely many PV fields. By an explicit calculation of the vacuum polarization tensor up to Λ^{-4} order, all the quadratic divergence exactly cancels in all orders. This result shows that the quadratic divergence is regularized by the hybrid regularization as well as the logarithmic divergence.

The divergence cancels order by order in Λ^{-4} and the cancellation mechanism is the same in all orders: the combination of (22) and (24) cancels with (23). We expect that this mechanism works in all the higher orders than Λ^{-4} e.g. in the order of Λ^{-8} and the quadratic divergence completely cancels out.

In our calculation, the higher derivative term for the ghost field, H , plays an important role in the cancellation of the quadratic divergence. Since any contribution of H does not appear in the superficial degree of divergence ω in (10), the necessity of H is unclear. So the simplest choice of the higher derivative term to improve the longitudinal part of the gauge propagator is the function f in the reference [8]. In such a case, the treatment of the non-local contribution is so problematic that the cancellation of the quadratic divergence is not shown by the calculation. H , however, is introduced instead of f at the beginning, such a difficulty does not arise and we can demonstrate the cancellation clearly.

We are also interested in the coefficient of the β -function with this regularization scheme. Since H gives the same effect with f and does not give any contribution to the logarithmic divergence in the order of Λ^0 we get the familiar value of the coefficient. We will discuss in detail the logarithmic divergence of this theory elsewhere [14].

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